# III. *On some Remarkable Relations which obtain among the Roots of the Four Squares into which a Number may be divided, as compared with the corresponding Roots of certain other Numbers. By the Right Hon. Sir* Frederick Pollock, *F.R.S., Lord Chief Baron.*

Received April 26,—Read May 20, 1858.

The property of numbers, which is the subject of this paper, first presented itself to my attention in the case of the odd squares 1, 9, 25, 49, &c. $(2n\mp1)^2$; any two adjoining odd squares may be divided (each of them) into 4 square numbers, whose roots will have this remarkable relation to each other: two of them will be identically the same; and as to the other two, one of them will be 2 less, and the other will be 2 more than the roots of the preceding or subsequent odd square; for example, 25 and 49 may be divided into squares, the roots of which being placed below, will appear thus:—

| | | | | | |
|---|---|---|---|---|---|
| | **25** | **49** | so | **49** | **81** |
| | −2, 1, 4, 2 | −4, 1, 4, 4 | | 0, 2, 3, 6 | −2, 2, 3, 8 |
| or thus | 0, 0, 3, 4 | −2, 0, 3, 6. | | | |

In comparing the roots of the adjoining odd squares, 2 roots (placed in the middle) are the same; of the others, one is 2 more, the other 2 less than the corresponding roots of the other.

The following Table presents the result of a comparison of the roots of all odd squares up to $27^2=729$:—

| | |
|---|---|
| **1** | **9** |
| 0, 0, 1, 0 | −2, 0, 1, 2 |
| **9** | **25** |
| 0, 1, 2, 2 | −2, 1, 2, 4 |
| **25** | **49** |
| −2, 1, 4, 2 | −4, 1, 4, 4 |
| 0, 0, 3, 4 | −2, 0, 3, 6 |
| **49** | **81** |
| 0, 2, 3, 6 | −2, 2, 3, 8 |
| **81** | **121** |
| 0, 1, 4, 8 | −2, 1, 4, 10 |
| −2, 4, 5, 6 | −4, 4, 5, 8 |
| −4, 0, 7, 4 | −6, 0, 7, 6 |

| | |
|---|---|
| **121** | **169** |
| −2, 2, 7, 8 | −4, 2, 7, 10 |
| **169** | **225** |
| 0, 3, 4, 12 | −2, 3, 4, 14 |
| −2, 4, 7, 10 | −4, 4, 7, 12 |
| −4, 5, 8, 8 | −6, 5, 8, 10 |
| −6, 4, 9, 6 | −8, 4, 9, 8 |
| **225** | **289** |
| −4, 3, 10, 10 | −6, 3, 10, 12 |
| −6, 5, 10, 8 | −8, 5, 10, 10 |
| **289** | **361** |
| −2, 5, 8, 14 | −4, 5, 8, 16 |
| −6, 3, 12, 10 | −8, 3, 12, 12 |

| | |
|---|---|
| **361** | **441** |
| 0, 1, 6, 18 | −2, 1, 6, 20 |
| −2, 1, 10, 16 | −4, 1, 10, 18 |
| −4, 7, 10, 14 | −6, 7, 10, 16 |
| −6, 9, 10, 12 | −8, 9, 10, 14 |
| **441** | **529** |
| −8, 8, 13, 12 | −10, 8, 13, 14 |
| −10, 4, 15, 10 | −12, 4, 15, 12 |

| | |
|---|---|
| **529** | **625** |
| −8, 10, 13, 14 | 10, 10, 13, 16 |
| −2, 5, 10, 20 | −4, 5, 10, 22 |
| **625** | **729** |
| −2, 4, 11, 22 | 4, 4, 11, 24 |
| &c. | &c. |

[*It is to be remembered, that throughout this paper the root of any square may be assumed to be positive or negative at pleasure, and* 0 *is considered as an even square.*]

This relation among the roots belongs also to the numbers resulting from the addition of any (the same) even number to the adjoining odd squares, and within certain limits (to be noticed presently) to the numbers resulting from the subtraction of the same even number: thus,

| | |
|---|---|
| **49−4=45** | **81−4=77** |
| −2, 3, 4, 4 | −4, 3, 4, 6 |
| 0, 0, 3, 6 | −2, 0, 3, 8 |
| **49−2=47** | **81−2=79** |
| −3, 2, 5, 3 | −5, 2, 5, 5 |
| **49** | **81** |
| 0, 2, 3, 6 | −2, 2, 3, 8 |
| **49+2=51** | **81+2=83** |
| −1, 3, 4, 5 | −3, 3, 4, 7 |
| **49+4=53** | **81+4=85** |
| 0, 1, 4, 6 | −2, 1, 4, 8 |
| &c. | &c. |

The addition may proceed indefinitely, but the subtraction has this limit: on deducting 30 from 49, and also from 81, the numbers become 19 and 51, which are terms in the series 1, 3, 9, 19, 33, 51, &c. ($2n^2+1$); and any 2 alternate terms of that series will become by continued addition adjoining odd squares.

Any 2 alternate terms of the series may be represented by $2.(n-1)^2+1$, and $2.(n+1)^2+1$, or $2n^2-4n+3$, and $2n^2+4n+3$; add $2n^2-2$ to each, and they become $(2n-1)^2$, and $(2n+1)^2$, that is, adjoining odd squares.

If, instead of the odd squares, a series of the even squares +1 be assumed, the adjoining terms will have similar properties; thus,

| | |
|---|---|
| **4+1=5** | **16+1=17** |
| 0, 0, 2, 1 | −2, 0, 2, 3 |
| +1, 0, 0, 2 | −1, 0, 0, 4 |

| | |
|---|---|
| **16+1=17** | **36+1=37** |
| $+1, 0, 0, 4$ | $-1, 0, 0, 6$ |
| **36+1=37** | **64+1=65** |
| $-1, 2, 4, 4$ | $-3, 2, 4, 6$ |
| &c. | &c. |

And each of these may in like manner be increased, or (subject to a similar limit) be diminished; these, however, are derived from the alternate terms of another series, 1, 5, 13, 25, &c. $(2n^2+2n+1)$, $(n-2)^2+(n-1)^2$, and $n^2+(n+1)^2$ will represent any 2 alternate terms of the series; and if to $2n^2-6n+5$, and also to $2n^2+2n+1$ there be added $2n^2-2n$, they become $4n^2-8n+5$, and $4n^2+1$, or $(2n-2)^2+1$, and $(2n)^2+1$, or, adjoining even squares $+1$.

The various examples of these or similar relations among the roots of the 4 squares into which numbers may be divided are endless; the increase and decrease of the variable roots is not always by 2; it may be by any other number. But instead of multiplying examples, it will be better to enter on the proof of what has been already stated, which will furnish the means of investigating every instance that can be produced. The proof depends upon,—

1st, a general property of all odd numbers which (as far as I am aware) has not hitherto been noticed; and

2ndly, a general theorem relating to odd numbers in arithmetical progression.

The property of odd numbers is this:—Every odd number may be divided into 4 squares, in such manner that 2 of the roots will be equal, 2 of them will differ by 1, 2 of them will differ by 2, 2 of them by 3, and so on, as far as the number is capable (from its magnitude) of having roots large enough to form the difference required. The difference may be algebraical, and result from one of the roots being considered as a negative quantity. For example, there is only one mode of dividing the number 23 (a number of the form $(8n+7)$) into 4 squares; these must be 1, 4, 9, 9, and their roots $\pm1$, $\pm2$, $\pm3$, $\pm3$:—

| | |
|---|---|
| 3 and 3 are equal, difference 0, | $-3$ and 3 differ by 6, |
| 3 and 2 differ by 1, | $-2$ and 3 differ by 5, |
| 3 and 1 differ by 2, | $-1$ and 3 differ by 4, |
| 2 and $-1$ differ by 3. | |

The differences of the roots may therefore be 0, 1, 2, 3, 4, 5 or 6; a greater difference than 6 is (in the number 23) impossible; the least numbers that would make a difference of 7 would be 3 and 4, and the sum of their squares would be 25.

The same form or mode of dividing a number into 4 squares does not always furnish every possible difference (as in the case of 23); thus 13 in one form has the roots 0, 0, 2, 3; these furnish as differences 0, 1, 2, 3, 5. To obtain the difference 4, the other form of dividing 13 into 4 squares (viz. $1^2$, $2^2$, $2^2$, $2^2$) must be resorted to, and then $-2$, $+2$ differ by 4: so 39 has two modes of dividing into 4 squares; in one the roots are

1, 2, 3, 5,—here are no equal roots; but the other mode has the roots 1, 1, 1, 6, which furnish two equal roots. Now it is well known that every odd number is of the form $a^2+b^2+2c^2$, that is, has 2 of its roots equal*; also it is easy to prove that every odd number will have 2 of its roots differing by 1, that is, will be of the form

$$a^2+b^2+c^2+(c+1)^2.$$

For any number of the form $4n+1$ may be composed of 3 squares (4 are not necessary); and as only one of them can be an odd square (for if there were 3 odd squares its form must be $4n+3$),

$$\therefore\ 4n+1=4a^2+4b^2+4c^2+4c+1,$$

and $n$ (any number) $=a^2+b^2+c^2+c$,

and $2n+1$ (any odd number)

$$=2a^2+2b^2+2c^2+2c+1=(a+b)^2+(a-b)^2+c^2+(c+1)^2;$$

but if 2 of the roots must be equal, and 2 of them must differ by 1, it will be shown that all the other differences must exist. Here I must request permission to introduce a new symbol, in order to denote a number so divided into squares as that 2 of its roots may have a certain difference; these roots I always make the exterior roots. I propose to place the index or number denoting the difference above the number to the left hand: thus ${}^{4}25$ denotes 25 with roots 0, 0, 3, 4 or $-2, 1, 4, 2$; ${}^{7}25$ may be $-3, 0, 0, 4$; ${}^{2n}(2n^2+1)$ may be $-n, 0, 1, n$.

In order to complete the proof, I must now call attention more particularly to the properties of the two series already mentioned.

The first is 1, 3, 9, 19, &c., general term $(2n^2+1)$; it increases by the numbers 2, 6, 10, 14, &c., whose sum is $(2n^2)$.

The second is 1, 5, 13, 25, &c., general term $(2n^2+2n+1)$; the increase is by the numbers 4, 8, 12, 16, &c., whose sum is $(2n^2+2n)$.

The first is the series of the least odd numbers whose roots may differ by the *even* numbers.

The second is the series of the least odd numbers whose roots may differ by the *odd* numbers. This is quite obvious: the number ${}^{2n}(2n^2+1)$ is $-n, 0, 1, n$; any other odd number with the same difference of roots must be greater, $-(n-1), 0, 1, n+1=2n^2+3$; so $(2n^2+2n+1)$, the difference of whose roots may be $(2n+1)$; $-n, 0, 0, n+1$ is the least number that can have that difference of roots.

These two series are mentioned as "*the two series*" during the rest of this paper. They may be formed together by increasing from 1, by the even numbers repeated (2, 2, 4, 4, 6, 6, &c.); thus, 1, 3, 5, 9, 13, 19, &c. This series has alternately the terms of each, but 1 belongs to both—as ${}^{0}1$ to the one, as ${}^{1}1$ to the other. Putting the symbol I have suggested above and the roots below, the first series will appear thus:

| | | | | | |
|---|---|---|---|---|---|
| ${}^{0}\mathbf{1}$ | ${}^{2}\mathbf{3}$ | ${}^{4}\mathbf{9}$ | ${}^{6}\mathbf{19}$ | &c. . . . . . | ${}^{2n}(2n^2+1)$ |
| 0, 0, 1, 0 | $-1, 0, 1, 1$ | $-2, 0, 1, 2$ | $-3, 0, 1, 3$ | | $-n, 0, 1, n$ |

* LEGENDRE, Théorie des Nombres, 1st edit. p. 186; 2nd edit. p. 394.

The middle roots are all the same; the exterior roots increase on the right and decrease on the left, by 1, at each step.

The other series will appear thus:

| | | | | | | |
|---|---|---|---|---|---|---|
| $^{1}1$ | $^{3}5$ | $^{5}13$ | $^{7}25$ | $^{9}41$ | &c. . . | $^{2n+1}(2n^2+2n+1)$ |
| $0, 0, 0, 1$ | $-1, 0, 0, 2$ | $-2, 0, 0, 3$ | $-3, 0, 0, 4$ | $-4, 0, 0, 5$ | | $-n, 0, 0, (n+1)$ |

Here also the middle roots are the same, and the exterior roots increase and decrease in the same manner, by 1, at each step; but if any odd number have two of its roots equal, it may be the first term of a series of the first kind; or if any odd number have two of its roots differing by 1, it may be the first term of a series of the second kind; any odd number may therefore be the first term of a series of either kind; thus—

| | | | | | |
|---|---|---|---|---|---|
| $^{0}23$ | $^{2}25$ | $^{4}31$ | $^{6}41$ | &c. . . . | $^{2n}(2n^2+23)$ |
| $+3, 1, 2, 3$ | $+2, 1, 2, 4$ | $+1, 1, 2, 5$ | $0, 1, 2, 6$ | | $-(n-3) 1, 2, (n+3)$ |

or

| | | | | | |
|---|---|---|---|---|---|
| $^{1}23$ | $^{3}27$ | $^{5}35$ | $^{7}47$ | &c. . . . | $^{2n+1}(2n^2+2n+23)$ |
| $+1, 3, 3, 2$ | $0, 3, 3, 3$ | $-1, 3, 3, 4$ | $-2, 3, 3, 5$ | | $(n-1), 3, 3, (n+2)$ |

The middle roots are the same; the exterior roots increase and decrease in the same way, and the terms of the two series increase by the same numbers, as if 1 were the first term of both; the difference between any two adjoining terms is the sum of their indices; if not adjoining, the difference between any two terms is the sum of their indices, plus twice the sum of the intermediate indices; the roots which compose any odd number to a given index may therefore be found thus:—Let 117 be the given number, and 12 the required index; then $^{0}a$, $^{2}b$, $^{4}c$, $^{6}d$, $^{8}e$, $^{10}f$, $^{12}117$ will represent the series whose seventh term is $^{12}117$: $0+12+2\times(2+4+6+8+10)=72$ is the difference between $a$ and 117; $a$ therefore $=117-72=45$; and the series with its roots will be

| | | | | | | |
|---|---|---|---|---|---|---|
| $^{0}45$ | $^{2}47$ | $^{4}53$ | $^{6}63$ | $^{8}77$ | $^{10}95$ | $^{12}117$ |
| $+2, 1, 6, 2$ | $+1, 1, 6, 3$ | $0, 1, 6, 4$ | $-1, 1, 6, 5$ | $-2, 1, 6, 6$ | $-3, 1, 6, 7$ | $-3, 1, 6, 8$ |

In the same manner $^{13}117$ may be discovered by the other series, and the result will be

| | | | | | | |
|---|---|---|---|---|---|---|
| $^{1}33$ | $^{3}37$ | $^{5}45$ | $^{7}57$ | $^{9}73$ | $^{11}93$ | $^{13}117$ |
| $+2, 2, 4, 3$ | $+1, 2, 4, 4$ | $0, 2, 4, 5$ | $-1, 2, 4, 6$ | $-2, 2, 4, 7$ | $-3, 2, 4, 8$ | $-4, 2, 4, 9$ |

What is stated above of 117 and the index 12, is obviously applicable to any other odd number and any other index; and it follows that if every odd number has two of its roots equal, and also two of its roots differing by 1, it will have other roots differing by 2, 3, 4, &c., as far as its magnitude will enable it.

There are various other modes of proving the same result. If it be required to find the roots which make $^{9}117$, begin with $^{1}117$, $^{3}121$, $^{5}129$, $^{7}141$, $^{9}157$; 157 is 40 more than 117; deduct 40 from every term, and the series becomes

| | | | | |
|---|---|---|---|---|
| $^{1}77$ | $^{3}81$ | $^{5}89$ | $^{7}101$ | $^{9}117$ |
| $+1, 6, 6, 2$ | | | | $-3, 6, 6, 6$ |

$^{1}77$ is $+1, 6, 6, 2$, and therefore $^{9}117$ is $-3, 6, 6, 6$. Obviously it is easy to throw the proof into a general algebraic form; the general property of odd numbers above stated may therefore be considered as established.

The theorem alluded to is this:—If any odd number of odd numbers, beginning with 13 (or any other number), be in arithmetical progression with a common difference of 4 [or any other number] (for the purpose of this example let the series be 13, 17, 21, 25, 29, 33, 37, with a common difference of 4), then if the common difference (4) be assumed as the index of the difference of roots to the middle term in the series, and the higher terms beyond the middle have as indices of the differences of their roots (4+1), (4+2), (4+3), &c. in succession, and the lower terms have as indices (4−1), (4−2), (4−3), &c., the series with the indices will be $^{1}13$, $^{2}17$, $^{3}21$, $^{4}25$, $^{5}29$, $^{6}33$, $^{7}37$; then if the terms less than the middle term be divided into four squares with exterior roots, having the differences indicated by the respective indices [which may be done by the property of odd numbers just proved], thus,

| $^{1}13$ | $^{2}17$ | $^{3}21$ | $^{4}25$ |
|---|---|---|---|
| 1, 2, 2, 2 | 0, 2, 3, 2 | −1, 0, 4, 2 | −2, 1, 4, 2 |
| 2, 0, 0, 3 | | +1, 0, 2, 4 | 0, 0, 3, 4 |

then the terms greater than the middle term will have this relation to the terms less than the middle term, the two terms next to the middle term will have their exterior roots,—one, less by 1, the other, greater by 1, than those of the other. The two terms next but one will have their exterior roots,—the one, less by 2, the other, greater by 2, and so on, increasing as the pairs of terms become more distant from the centre; and all the pairs of terms (equidistant from the middle term) will respectively have the same middle roots; thus

| $^{4}25$ | $^{5}29$ | $^{6}33$ | $^{7}37$ |
|---|---|---|---|
| −2, 1, 4, 2 | 0, 0, 2, 5 | −2, 2, 3, 4 | −2, 2, 2, 5 |
| | | | −1, 0, 0, 6 |

Comparing them, the result is as stated above; the difference between the exterior roots of the respective pairs is half the difference between the respective indices.

The algebraic proof of this theorem is very easy. Let there be an arithmetic series with $n$ as a middle term and $p$ as a common difference; then the series with the indices of the differences of roots will be

$$\ldots\ldots \quad {}^{p-2}(n-2p), \quad {}^{p-1}(n-p), \quad {}^{p}n, \quad {}^{p+1}(n+p), \quad {}^{p+2}(n+2p), \text{ \&c.};$$

and any two terms equidistant from the middle term may be represented by $(n-mp)$, $(n+mp)$, and their indices will be $(p-m)$, $(p+m)$; then if ${}^{p-m}(n-mp)$ have its middle roots $r$, $s$, and the first of its exterior roots $a$, the other exterior root must be $a+p-m$ in order that the difference of the roots may correspond with the index and equal $(p-m)$; then the corresponding roots of $n+mp$ will be, according to the theorem, $(a-m), r, s, a+p$. If the first set of roots be squared, the sum is

$$a^2+r^2+s^2+(a+p-m)^2=a^2+r^2+s^2+(a+p)^2+m^2-2am-2pm;$$

if the second set of roots be squared, the sum is $a^2-2am+m^2+r^2+s^2+(a+p)^2$, which is $2pm$ more than the former; if, therefore, the first sum of squares equals $n-mp$, the second will equal $n+mp$; therefore the pair of terms that are at the distance $(m)$ from the centre will have their middle roots the same, and their exterior roots one less by $m$, the other greater by $m$ than those of the other.

In this proof $p$, the common difference, may be odd or even, and $n$, the middle term, may be odd or even; thus

| | | | | |
|---|---|---|---|---|
| $^{2}\mathbf{2}$ | $^{3}\mathbf{6}$ | $^{4}\mathbf{10}$ | $^{5}\mathbf{14}$ | $^{6}\mathbf{18}$ |
| $-1,0,0,1$ | $-1,0,1,2$ | $-2,1,1,2$ | $-2,0,1,3$ | $-3,0,0,3$ |

is a series composed of even numbers, all of which obey the theorem; but frequently an even number is not so divisible as to form the required difference. To form every difference is a property which belongs universally to odd numbers only, not to even numbers; the common difference may be only 1; the numbers from 25 to 41 are all (both odd and even) divisible into four squares, whose roots conform to the theorem, 33 being the middle term.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $^{-7}\mathbf{25}$ | $^{-6}\mathbf{26}$ | $^{-5}\mathbf{27}$ | $^{-4}\mathbf{28}$ | $^{-3}\mathbf{29}$ | $^{-2}\mathbf{30}$ | $^{-1}\mathbf{31}$ | $^{0}\mathbf{32}$ | |
| $4,0,0-3$ | $5,0,0,-1$ | $5,1,1,0$ | $2,2,4,-2$ | $5,0,0,2$ | $3,2,4,1$ | $3,3,3,2$ | $0,4,4,0$ | |
| | | | | | | | | $^{1}\mathbf{33}$ |
| | | | | | | | | $0,4,4,1$ |
| $^{9}\mathbf{41}$ | $^{8}\mathbf{40}$ | $^{7}\mathbf{39}$ | $^{6}\mathbf{38}$ | $^{5}\mathbf{37}$ | $^{4}\mathbf{36}$ | $^{3}\mathbf{35}$ | $^{2}\mathbf{34}$ | |
| $-4,0,0,5$ | $-2,0,0,6$ | $-1,1,1,6$ | $-3,2,4,3$ | $+1,0,0,6$ | $0,2,4,4$ | $1,3,3,4$ | $-1,4,4,1$ | |

To apply this theorem as a proof of the matters stated in the beginning of the paper, all the examples, whether of the odd squares, or of the even squares $+1$, or those numbers increased or decreased, may be made terms in an arithmetic series; $4n^2-4n+1$ and $4n^2+4n+1$ (which represent any 2 adjoining odd squares) have a difference of $8n$, which is divisible by 4, and therefore they may form terms in an arithmetical series; thus

$$4n^2-4n+1,\quad 4n^2-2n+1,\quad 4n^2+1,\quad 4n^2+2n+1,\quad 4n^2+4n+1,$$

the common difference being $2n$, and the odd squares will be two places from the middle term; their exterior roots will therefore be greater and less by 2. So any 2 adjoining odd squares, $4n^2$ and $4n^2+8n+4$, differ by $8n+4$, which is divisible by 4; and the adjoining even squares, $+1$, may in like manner be made terms in an arithmetic series.

I propose now to apply the theorem generally to other instances of odd numbers having 2 roots equal and the other 2 roots differing by any number whatever, the one root being greater, the other less by that number; for example, the alternate odd squares may be divided into 4 squares in such manner that 2 roots of the one may equal 2 roots of the other, and the differences of the remaining roots will be 4:

| | |
|---|---|
| $^{1}\mathbf{9}$ | $^{9}\mathbf{49}$ |
| $0,2,2,1$ | $-4,2,2,5$ |
| $+1,0,2,2$ | $-3,0,2,6$ |

| | |
|---|---|
| $^{3}$**25** | $^{11}$**81** |
| $0, 0, 4, 3$ | $-4, 0, 4, 7$ |
| $-1, 2, 4, 2$ | $-5, 2, 4, 6$ |

so the odd squares taken every third term, as

| | |
|---|---|
| **9** | **81** |
| $+2, 0, 1, 2$ | $-4, 0, 1, 8$ |
| **25** | **121** |
| $+2, 1, 2, 4$ | $-4, 1, 2, 10$ |

present a difference of 6 for the exterior roots.

The theorem affords a solution of all these and every other instance. For in the case of any arithmetical progression of odd numbers having an odd number of terms, the terms of which have been indexed as directed (by making the common difference the index of the middle term, &c.), it will be found that all the pairs of terms equidistant from the middle term are derived from 2 terms of one, or other, of the 2 series above-mentioned by adding the same even number to both; if the indices be even, they are derived from terms in the series 1, 3, 9, 19, &c.; if the indices be odd, the terms are derived from the series 1, 5, 13, 25, &c., and the terms in the series may immediately be found, as they are the terms having the same indices as the pair of terms in the arithmetical progression. Let

$$^{0}5,\quad ^{1}9,\quad ^{2}13,\quad ^{3}17,\quad ^{4}21,\quad ^{5}25,\quad ^{6}29,\quad ^{7}33,\quad ^{8}37$$

be an arithmetic series (with a common difference 4 and 21 the middle term) indexed according to the theorem; place under each term that term in either of the two series which has the index of the term in the arithmetical series; thus

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $^{0}5$ | $^{1}9$ | $^{2}13$ | $^{3}17$ | $^{4}21$ | $^{5}25$ | $^{6}29$ | $^{7}33$ | $^{8}37$ |
| $^{0}1$ | $^{1}1$ | $^{2}3$ | $^{3}5$ | $^{4}9$ | $^{5}13$ | $^{6}19$ | $^{7}25$ | $^{8}33$ |
| 4 | 8 | 10 | 12 | 12 | 12 | 10 | 8 | 4 |

Deducting the one from the other, it is obvious that the terms equidistant from 21 are derived from terms of the two series, by adding to them the same number; if the terms in the 2 series be adjacent, the difference of the exterior roots will be 1, if alternate, 2, and so on; if the terms in the 2 series be $m$ places distant, the differences of the exterior roots will be $m$: this may be shown generally in an algebraic form, thus.

Let the middle term of an arithmetic series be $n$ and the common difference be an even number $2m$, the series with its indices will be

$$^{2m-2}(n-6m),\ ^{2m-2}(n-4m),\ ^{2m-1}(n-2m),\ ^{2m}n,\ ^{2m+1}(n+2m),\ ^{2m+2}(n+4m),\ ^{2m+3}(n+6m),\ \&c.$$

and the terms of the 2 series to be placed under each term will be

$$\&c.\ (2m^2-4m+3),\ (2m^2-2m+1),\ (2m^2+1),\ (2m^2+2m+1),\ (2m^2+4m+3),\ \&c.;$$

deducting the one from the other, the remainders will be

$$n-(2m^2+5),\ \big(n-(2m^2+3)\big),\ \big(n-(2m^2+1)\big),\ n-(2m^2+1),\ n-(2m^2+1),\ n-(2m^2+3),\ n-(2m^2+5),$$

where obviously the remainders equidistant from the middle term are equal to one another.

Now the difference between the adjoining terms of either of the 2 series will always be divisible by 2, the difference between the alternate terms will be divisible by 4, the difference between 2 terms that are $m$ apart will be divisible by $2m$.

For the $n$th term of the series 1, 3, 9, 19, &c. is $2n^2+1$, and the $(n+m)$th term is $2n^2+4nm+2m^2+1$, and their difference, $4nm+2m^2$, is divisible by $2m$. So in the other series (1, 5, 13, 25, &c.) the $n$th term is $2n^2+2n+1$, and the $(n+m)$th term is

$$(n+m)^2+(n+m+1)^2=2n^2+2n+1+2m^2+4mn+2m;$$

and their difference, $2m^2+4mn+2m$, is also divisible by $m$. An arithmetical series may therefore always be formed, which will give the required difference of the roots according to the distance of the pair of terms from the middle term.

The general result therefore is, that if any 2 odd numbers be assumed, they will either have this relation to each other of the roots of the 4 square numbers into which they may be divided, or a third odd number may be found, which will connect them together by having that relation to each.

If the 2 odd numbers be the result of an addition of the same even number to any 2 terms of either of the two series, they will have this relation of the roots, and the difference of the exterior roots will depend upon the distance of the 2 terms from each other; and conversely, if any 2 odd numbers have this relation of the roots, they are derived from 2 terms of the same series by the addition of the same even number to both; but if the 2 odd numbers have not this relation of their roots to each other, then a third odd number may be found having that relation to each of them.

Assume any 2 odd numbers as 13 and 105, deduct 12 from each of them so as to reduce the smaller to 1, and the other to 93, the next term in either series less than 93 is 85, $85+8=93$. Select that term in the series to which 85 belongs, which by the addition of 8 becomes a term in the other series, this is 25, $25+8=33$, and 33 is the number which connects 1 with 93; for

| | | |
|---|---|---|
| $^{0}\mathbf{1}$ | and | $^{8}\mathbf{33}$ |
| 0, 0, 1, 0 | | −4, 0, 1, 4 |

have the relation which appears from their roots, and

| | | |
|---|---|---|
| $^{7}\mathbf{33}$ | and | $^{13}\mathbf{93}$ |
| −3, 2, 2, 4 | | −6, 2, 2, 7 |
| −2, 0, 2, 5 | | −5, 0, 2, 8 |

have a similar relation in two ways; then add 12 to 1, to 33, and to 93, and 13, 45 and 105 will have this relation among their roots:

| $^{0}\mathbf{13}$ | $^{8}\mathbf{45}$ | $^{7}\mathbf{45}$ | $^{13}\mathbf{105}$ |
|---|---|---|---|
| +2, 1, 2, 2 | −2, 1, 2, 6 | −3, 2, 4, 4 | −6, 2, 4, 7 |
| 0, 2, 3, 0 | −4, 2, 3, 4 | −2, 0, 4, 5 | −5, 0, 4, 8 |
| | | −1, 2, 2, 6 | −4, 2, 2, 9 |

It is obvious, that what is done with these numbers may be done with any other odd numbers, and it would be superfluous to give an algebraic proof.

This relation of all odd numbers to each other has not (as far as I am aware) been remarked before; but it has occurred to me that possibly it may form part of the "mysteries of numbers" alluded to by FERMAT in that remarkable passage in which his theorem of the polygonal numbers was announced in a note to an edition of Diophantus, published after his death, p. 180*. The mysterious properties of numbers referred to by FERMAT must (of course) be connected with the theorem he was announcing; indeed he expressly refers to them as the source of his demonstration.

## POSTSCRIPT.—May 20, 1858.

Since this paper was sent to the Society, some other theorems of a similar kind have occurred to me, in which the terms of a series (not arithmetical of the 1st order) have a similar relation with regard to the roots of the 4 squares into which they may be divided, that is, those which are equidistant from the middle (if the number of terms be even), or from the middle term (if the number of terms be odd), have the middle roots the same, and the exterior roots have an arithmetical relation to each other, varying with the distance from the centre, viz. the one being less and the other greater by the same quantity.

Thus, if any number of terms (exceeding 3) of either of the 2 series above-mentioned, viz. 1, 3, 9, 19, &c. ($2n^2+1$), or 1, 5, 13, 25, &c. ($2n^2+2n+1$), and, beginning with the 1st term, the successive differences of the terms be added "*inverso ordine*," a new series will be obtained possessing the property in question; thus the first seven terms of the 1st series are $^{0}1$, $^{2}3$, $^{4}9$, $^{6}19$, $^{8}33$, $^{10}51$, $^{12}73$; the differences are 2, 6, 10, 14, 18, 22; if the differences be added "*inverso ordine*," beginning with 22 instead of 2, the series becomes 1, 23, 41, 55, 65, 71, 73, each term of which may be divided into 4 squares, whose roots will be as follows:—

| $^{0}\mathbf{1}$ | $^{2}\mathbf{23}$ | $^{4}\mathbf{41}$ | $^{6}\mathbf{55}$ | $^{8}\mathbf{65}$ | $^{10}\mathbf{71}$ | $^{12}\mathbf{73}$ |
|---|---|---|---|---|---|---|
| 0, 0, 1, 0 | +1, 2, 3, 3 | 0, 3, 4, 4 | −3, 1, 6, 3 | −2, 3, 4, 6 | −3, 2, 3, 7 | 6, 0, 1, 6 |
| | | +2, 0, 1, 6 | −1, 2, 5, 5 | 0, 0, 1, 8 | | |
| | | | +1, 1, 2, 7 | | | |

Here obviously the result is as stated above; 55 is the middle term; the terms equidistant from it have the same middle roots, and the difference between the other roots increases according to the distance from the middle term being 2, 4, 6. If 8 terms be so treated, there is no middle term; the result is similar, but the successive differences are 1, 3, 5, 7.

The other series, 1, 5, 13, 25, &c., gives a similar result.

The reason of these results is that the equidistant terms are always equal to the original corresponding term in the series, increased by the same number; thus,

$$41 \text{ and } 65 = 9+32 \text{ and } 33+32 \text{ respectively.}$$

* See LEGENDRE, Théorie des Nombres, 1st edit. p. 187.

There are certain numbers, which, added to the terms of the series in its ordinary state, convert it into a series whose differences have been added "*inverso ordine.*"

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| If the number of terms be 3, add | 0 | 4 | 0 | | | | | |
| If 4, add . . . . . . . . . | 0 | 8 | 8 | 0 | | | | |
| If 5, add . . . . . . . . . | 0 | 12 | 16 | 12 | 0 | | | |
| If 6, add . . . . . . . . . | 0 | 16 | 24 | 24 | 16 | 0 | | |
| If 7, add . . . . . . . . . | 0 | 20 | 32 | 36 | 32 | 20 | 0 | |
| If 8, add . . . . . . . . . | 0 | 24 | 40 | 48 | 48 | 40 | 24 | 0 |
| | | | &c. | | | &c. | | |

The law of the formation of these numbers is obvious.

These numbers apply to both series, and to any consecutive terms in either; that is, [*e. g.*] 0, 12, 16, 12, 0, added to any 5 consecutive terms of either series, converts them into 5 terms whose differences have been added "*inverso ordine;*" and what is still more remarkable, the middle roots of the first eight (or indeed $n$) terms having their differences added "*inverso ordine,*" are the middle roots which answer for any eight [or $n$] consecutive terms, whose differences have been added "*inverso ordine*" through the unlimited extent of the whole series. Thus if the first 8 terms be formed with the differences added "*inverso ordine,*"

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **1** | **5** | **13** | **25** | **41** | **61** | **85** | **113** |
| add | 0 | 24 | 40 | 48 | 48 | 40 | 24 | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $^{1}$**1** | $^{3}$**29** | $^{5}$**53** | $^{7}$**73** | $^{9}$**89** | $^{11}$**101** | $^{13}$**109** | $^{15}$**113** |
| 0,0,0,1 | 0,2,4,3 | −2,2,6,3 | −1,0,6,6 | −2,0,6,7 | −5,2,6,6 | −5,2,4,8 | −7,0,0,8 |
| | +2,0,0,5 | −1,0,6,4 | +1,2,2,8 | 0,2,2,9 | −4,0,6,7 | −3,0,0,10 | |
| | | +1,0,4,6 | | | −2,0,4,9 | | |
| | | +2,0,0,7 | | | −1,0,0,10 | | |

the indices and the roots of the squares will be as above.

Now take the 8 consecutive terms, beginning with 181:—

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **181** | **221** | **265** | **313** | **365** | **421** | **481** | **545** |
| add | 0 | 24 | 40 | 48 | 48 | 40 | 24 | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $^{19}$**181** | $^{21}$**245** | $^{23}$**305** | $^{25}$**361** | $^{27}$**413** | $^{29}$**461** | $^{31}$**505** | $^{33}$**545** |
| −9,0,0,10 | −9,2,4,12 | −11,2,6,12 | −10,0,6,15 | −11,0,6,16 | −14,2,6,15 | −14,2,4,17 | −16,0,0,17 |
| | −7,0,0,14 | −10,0,6,13 | − 8,2,2,17 | − 9,2,2,18 | −13,0,6,16 | −12,0,0,19 | |
| | | − 8,0,4,15 | | | −11,0,4,18 | | |
| | | − 7,0,0,16 | | | −10,0,0,19 | | |

the middle roots are the same for both.

These last matters add weight to the suggestion already made, that the properties of numbers referred to are connected with the "mysterious and abstruse" properties alluded to by FERMAT, as enabling him to prove the theorem he announced of the polygonal numbers.